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Generalized Quasi-Variational Inequalities in Locally Convex Topological Vector Spaces*

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Let E be a Hausdorff topological vector space and $X \subset E$ an arbitrary nonempty set. Denote by E' the dual space of E and the pairing between E' and E by $\langle w, x \rangle$ for $w \in E'$ and $x \in E$. Given a point-to-set map $S: X \to 2^X$ and a point-to-set map $T: X \to 2^{E'}$, the generalized quasi-variational inequality problem (GQVI) is to find a point $\hat{y} \in S(\hat{y})$ and a point $\hat{u} \in T(\hat{y})$ such that $\operatorname{Re}\langle \hat{u}, \hat{y} - x \rangle \leqslant 0$ for all $x \in S(\hat{y})$. By using the Ky Fan minimax principle or its generalized version as a tool, some general theorems on solutions of the GQVI in locally convex Hausdorff topological vector spaces are obtained which include a fixed point theorem due to Ky Fan and I. L. Glicksberg, and two different multivalued versions of the Hartman–Stampacchia variational inequality.

1

Let E be a Hausdorff topological vector space, $X \subset E$ an arbitrary nonempty set and 2^X the collection of all subsets of X. We shall denote by E' the dual space of E (i.e., the vector space of all continuous linear functionals on E). We denote the pairing between E' and E by $\langle w, x \rangle$ for w in E' and x in E. Given a (point-to-set) map $S: X \to 2^X$ and a (point-to-point) map $T: X \to E'$, the quasi-variational inequality problem (QVI) is to find a point $\hat{y} \in S(\hat{y})$ such that $\operatorname{Re}\langle T(\hat{y}), \hat{y} - x \rangle \leqslant 0$ for all $x \in S(\hat{y})$. The QVI was introduced by Bensousson and Lions in 1973 (see, e.g., [3]) in

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409/108/2-3

connection with *impulse control*. A recent work concerning the QVI may be found in Mosco [11]. If we consider a point-to-set map $T: X \to 2^{E'}$, then the generalized quasi-variational inequality problem (GQVI) is to find a point $\hat{y} \in S(\hat{y})$ and a point $\hat{u} \in T(\hat{y})$ such that $\operatorname{Re}\langle \hat{u}, \hat{y} - x \rangle \leqslant 0$ for all $x \in S(\hat{y})$ (see [5]).

In the present paper we shall give some general theorems on solutions of the GQVI. Our basic tool is the Ky Fan minimax principle [7] or the following generalized version due to Yen [15].

THEOREM A. *Let X be a nonempty compact convex set in a Hausdorff topological vector space E. Let ϕ and ψ be two real-valued functions on $X \times X$ having the following properties*:

(a) *$\phi \leqslant \psi$ on $X \times X$ and $\psi(x, x) \leqslant 0$ for all $x \in X$*;

(b) *For each fixed $x \in X$, $\phi(x, y)$ is a lower semicontinuous function of y on X*;

(c) *For each fixed $y \in X$, $\psi(x, y)$ is a quasi-concave function of x on X.*

Then there exists a point $\hat{y} \in X$ such that $\phi(x, \hat{y}) \leqslant 0$ for all $x \in X$.

Here, a real-valued function ψ defined on a convex set X is said to be *quasi-concave* if for every real number λ, the set $\{x \in X : \psi(x) > \lambda\}$ is convex.

2

Let X be any nonempty subset of a Hausdorff topological vector space E. A set-valued map $T: X \to 2^{E'}$ is said to be monotone on X [4, p. 79] if for all x and y in X, each u in $T(x)$, and each w in $T(y)$, $\operatorname{Re}\langle w - u, y - x \rangle \geqslant 0$. We need the following two kinds of continuity for set-valued maps. Let M and N be topological spaces, and let $\Gamma: M \to 2^N$ be a set-valued map. We say that Γ is *upper semicontinuous* at $x_0 \in M$ [2, p. 109] if for each open set G with $\Gamma(x_0) \subset G$ there exists a neighborhood $N(x_0)$ of x_0 such that if $x \in N(x_0)$, then $\Gamma(x) \subseteq G$; Γ is upper semicontinuous on M if it is upper semicontinuous at each point of M. Also, Γ is lower semicontinuous at $x_0 \in M$ [2, p. 109] if for each open set G with $\Gamma(x_0) \cap G \neq \varnothing$ there is a neighborhood $N(x_0)$ of x_0 such that if $x \in N(x_0)$, then $\Gamma(x) \cap G \neq \varnothing$; Γ is lower semicontinuous on M if it is lower semicontinuous at each point of M. Moreover, Γ is said to be continuous on M if it is both upper semicontinuous and lower semicontinuous on M.

Our proofs of Theorems 1 and 3 require the following lemma.

LEMMA 1. *Let E be a Hausdorff topological vector space, $X \subset E$ be nonempty and $S: X \to 2^E$ be upper semicontinuous such that for each $x \in X$, $S(x)$*

is nonempty and bounded. Then for $p \in E'$ *the map* $f_p: X \to \mathbb{R}$ *defined by* $f_p(y) := \sup_{x \in S(y)} \operatorname{Re}\langle p, x \rangle$ *is upper semicontinuous.*

Proof. Let $y_0 \in X$ and $\varepsilon > 0$ be given. Let

$$U_\varepsilon := \{x \in E: |\langle p, x \rangle| < \varepsilon/2\};$$

then U_ε is an open neighborhood of 0. As $S(y_0) + U_\varepsilon$ is an open set containing $S(y_0)$, by upper semicontinuity of S at y_0, there exists a neighborhood $N(y_0)$ of y_0 in X such that if $y \in N(y_0)$ then $S(y) \subset S(y_0) + U_\varepsilon$. Thus, for each $y \in N(y_0)$,

$$\begin{aligned} f_p(y) &= \sup_{x \in S(y)} \operatorname{Re}\langle p, x \rangle \\ &\leqslant \sup_{x \in S(y_0) + U_\varepsilon} \operatorname{Re}\langle p, x \rangle \\ &\leqslant \sup_{x \in S(y_0)} \operatorname{Re}\langle p, x \rangle + \sup_{x \in U_\varepsilon} \operatorname{Re}\langle p, x \rangle \\ &< f_p(y_0) + \varepsilon. \end{aligned}$$

Hence f_p is upper semicontinuous and the proof is completed. ∎

THEOREM 1. *Let E be a locally convex Hausdorff topological vector space* and *X be a nonempty compact convex subset of E. Let $S: X \to 2^X$ be upper semicontinuous such that for each $x \in X$, $S(x)$ is a nonempty closed convex subset of X, and let $T: X \to 2^{E'}$ be monotone such that for all $x \in X$, $T(x)$ is a nonempty subset of E' and for each one-dimensional flat $L \subset E$, $T|L \cap X$ is lower semicontinuous from the topology of E to the weak*-topology $\sigma(E', E)$ of E'. Suppose further that the set $\Sigma_1 := \{y \in X: \sup_{x \in S(y)} \sup_{u \in T(x)} \operatorname{Re}\langle u, y - x \rangle > 0\}$ is open* in *X. Then there exists a point $\hat{y} \in X$ such that*

(i) $\hat{y} \in S(\hat{y})$ *and*

(ii) $\sup_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x \rangle \leqslant 0$ *for all* $x \in S(\hat{y})$.

Proof. We divide the proof into two steps:

Step 1. *There exists a point* $\hat{y} \in X$ *such that* $\hat{y} \in S(\hat{y})$ *and* $\sup_{u \in T(x)} \operatorname{Re}\langle u, \hat{y} - x \rangle \leqslant 0$ *for all* $x \in S(\hat{y})$.

Suppose the assertion were false. Then for all $y \in X$, either $y \notin S(y)$ or there exists a point $x \in S(y)$ such that $\sup_{u \in T(x)} \operatorname{Re}\langle u, y - x \rangle > 0$. Observe that whenever $y \notin S(y)$, there exists $p \in E'$ such that

$$\operatorname{Re}\langle p, y \rangle - \sup_{x \in S(y)} \operatorname{Re}\langle p, x \rangle > 0$$

by applying the Hahn-Banach separation theorem. For each $y \in X$, we set

$$\alpha(y) := \sup_{x \in S(y)} \sup_{u \in T(x)} \operatorname{Re}\langle u, y-x\rangle.$$

Let

$$V_0 := \{y \in X : \alpha(y) > 0\}.$$

For each $p \in E'$, we set

$$V(p) := \{y \in X : \operatorname{Re}\langle p, y\rangle - \sup_{x \in S(y)} \operatorname{Re}\langle p, x\rangle > 0\}.$$

Then $X = V_0 \cup \bigcup_{p \in E'} V(p)$. By hypothesis, V_0 is open in X. By Lemma 1, $V(p)$ is open in X for each $p \in E'$. Since X is compact, there exist $p_1, ..., p_n \in E'$ such that $X = V_0 \cup \bigcup_{i=1}^n V(p_i)$ and a continuous partition of unity $\{\beta_0, \beta_1, ..., \beta_n\}$ subordinated to the covering $\{V_0, V(p_1), ..., V(p_n)\}$, that is, $\beta_0, \beta_1, ..., \beta_n$ are continuous nonnegative real-valued functions on X such that β_0 vanishes on $X \setminus V_0$ and for each $1 \leqslant i \leqslant n$, β_i vanishes on $X \setminus V(p_i)$ and $\sum_{i=0}^n \beta_i(x) = 1$ for all $x \in X$.

Define $\phi, \psi : X \times X \to \mathbb{R}$ by setting

$$\phi(x, y) := \beta_0(y) \sup_{u \in T(x)} \operatorname{Re}\langle u, y-x\rangle + \sum_{i=1}^n \beta_i(y) \operatorname{Re}\langle p_i, y-x\rangle,$$

$$\psi(x, y) := \beta_0(y) \inf_{w \in T(y)} \operatorname{Re}\langle w, y-x\rangle + \sum_{i=1}^n \beta_i(y) \operatorname{Re}\langle p_i, y-x\rangle.$$

By monotonicity of T, we have

$$\sup_{u \in T(x)} \operatorname{Re}\langle u, y-x\rangle \leqslant \inf_{w \in T(y)} \operatorname{Re}\langle w, y-x\rangle \qquad \text{for all } x, y \in X.$$

It follows that $\phi \leqslant \psi$ on $X \times X$. Clearly $\psi(x, x) = 0$ for all $x \in X$. For each fixed $x \in X$, since $\beta_i (i = 0, 1, ..., n)$ are continuous *nonnegative* functions of y on X and $\sup_{u \in T(x)} \operatorname{Re}\langle u, y-x\rangle$, $\operatorname{Re}\langle p_i, y-x\rangle$ $(i = 1, ..., n)$ are lower semicontinuous functions of y on X, by Lemma 3 in [13, p. 177], $y \to \phi(x, y)$ is lower semicontinuous on X. Furthermore, for each fixed $y \in X$, $x \to \psi(x, y)$ is quasi-concave. Hence, all the conditions of Theorem A are satisfied, so that there exists a point $\hat{y} \in X$ such that $\phi(x, \hat{y}) \leqslant 0$ for all $x \in X$; that is,

$$\beta_0(\hat{y}) \sup_{u \in T(x)} \operatorname{Re}\langle u, \hat{y}-x\rangle + \sum_{i=1}^n \beta_i(\hat{y}_i) \operatorname{Re}\langle p_i, \hat{y}-x\rangle \leqslant 0 \qquad \text{for all } x \in X. \quad (*)$$

Since $\{\beta_0, \beta_1, ..., \beta_n\}$ is a partition of unity, $\beta_i(\hat{y}) > 0$ for at least one index $i \in \{0, 1, ..., n\}$. Choose any $\hat{x} \in S(\hat{y})$ such that

$$\sup_{u \in T(\hat{x})} \operatorname{Re}\langle u, \hat{y} - \hat{x}\rangle \geqslant \frac{\alpha(\hat{y})}{2} \qquad \text{whenever } \alpha(\hat{y}) > 0.$$

If $\beta_0(\hat{y}) > 0$, then $\hat{y} \in V_0$ so that $\alpha(\hat{y}) > 0$. Hence,

$$\sup_{u \in T(\hat{x})} \operatorname{Re}\langle u, \hat{y} - \hat{x}\rangle \geqslant \frac{\alpha(\hat{y})}{2} > 0.$$

If $\beta_i(\hat{y}) > 0$ for $i = 1, ..., n$, then $\hat{y} \in V(p_i)$ and hence

$$\operatorname{Re}\langle p_i, \hat{y}\rangle > \sup_{x \in S(y)} \operatorname{Re}\langle p_i, x\rangle \geqslant \operatorname{Re}\langle p_i, \hat{x}\rangle$$

so that $\operatorname{Re}\langle p_i, \hat{y} - \hat{x}\rangle > 0$. It follows that

$$\beta_0(\hat{y}) \sup_{u \in T(x)} \operatorname{Re}\langle u, \hat{y} - \hat{x}\rangle + \sum_{i=1}^{n} \beta_i(\hat{y}) \operatorname{Re}\langle p_i, \hat{y} - \hat{x}\rangle > 0,$$

contradicting $(*)$. This contradiction proves Step 1.

Step 2. $\sup_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x\rangle \leqslant 0$ *for all* $x \in S(\hat{y})$.

Let $x \in S(\hat{y})$ be arbitrarily fixed and let $z_t := tx + (1-t)\hat{y} \equiv \hat{y} - t(\hat{y} - x)$ for $t \in [0, 1]$. As $S(\hat{y})$ is convex, we have $z_t \in S(\hat{y})$ for $t \in [0, 1]$. Therefore by Step 1, we have

$$\sup_{u \in T(z_t)} \operatorname{Re}\langle u, \hat{y} - z_t\rangle \leqslant 0 \qquad \text{for all } t \in [0, 1],$$

and it follows that

$$\sup_{u \in T(z_t)} \operatorname{Re}\langle u, \hat{y} - x\rangle \leqslant 0 \qquad \text{for all } t \in (0, 1]. \tag{$**$}$$

Let $w_0 \in T(\hat{y})$ be arbitrarily fixed. For each $\varepsilon > 0$, let

$$U_{w_0} := \{w \in E' : |\langle w_0 - w, \hat{y} - x\rangle| < \varepsilon\};$$

then U_{w_0} is a $\sigma(E', E)$-neighborhood of w_0. Since $T | L \cap X$ is lower semicontinuous, where $L := \{z_t : t \in [0, 1]\}$, and $U_{w_0} \cap T(\hat{y}) \neq \varnothing$, there exists a neighborhood $N(\hat{y})$ of $\hat{y}$ in L such that if $z \in N(\hat{y})$ then $T(z) \cap U_{w_0} \neq \varnothing$. But then there exists $\delta \in (0, 1)$ such that $z_t \in N(\hat{y})$ for all $t \in (0, \delta)$. Fix any $t \in (0, \delta)$ and $u \in T(z_t) \cap U_{w_0}$, we have

$$|\langle w_0 - u, \hat{y} - x\rangle| < \varepsilon.$$

This implies

$$\mathrm{Re}\langle w_0, \hat{y}-x\rangle < \mathrm{Re}\langle u, \hat{y}-x\rangle + \varepsilon.$$

By (**), we have $\mathrm{Re}\langle w_0, \hat{y}-x\rangle < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\mathrm{Re}\langle w_0, \hat{y}-x\rangle \leqslant 0$. As $w_0 \in T(\hat{y})$ is arbitrary,

$$\sup_{w \in T(\hat{y})} \mathrm{Re}\langle w, \hat{y}-x\rangle \leqslant 0 \qquad \text{for all } x \in S(\hat{y}).$$

This concludes the proof of our theorem. ∎

In the first step of our proof, we follow the argument of Aubin [1, pp. 373–374]. In the second step of our proof, we use the argument of Shih and Tan [12] and Tan [14].

When $T \equiv 0$, Theorem 1 gives the well-known Fan–Glicksberg fixed point theorem [6, 8].

COROLLARY 1 (Fan and Glicksberg). *Let E be a locally convex Hausdorff topological vector space and X a nonempty compact convex set in E. Let $S: X \to 2^X$ be upper semicontinuous such that for each $x \in X$, $S(x)$ is a nonempty closed convex subset of X. Then there exists a point $\hat{x} \in X$ such that $\hat{x} \in S(\hat{x})$.*

We shall now observe that in Theorem 1, the interaction between the maps S and T (namely, Σ_1 is open in X) can be achieved by imposing additional continuity conditions on S and T.

THEOREM 2. *Let E be a locally convex Hausdorff topological vector space and X be a nonempty compact convex subset of E. Let $S: X \to 2^X$ be continuous such that for each $x \in X$, $S(x)$ is a nonempty closed convex subset of X, and $T: X \to 2^{E'}$ be monotone such that for each $x \in X$, $T(x)$ is a nonempty subset of E' and T is lower semicontinuous from the relative topology of X to the strong topology of E'. Then there exists a point $\hat{y} \in X$ such that*

(i) $\hat{y} \in S(\hat{y})$ *and*

(ii) $\sup_{w \in T(\hat{y})} \mathrm{Re}\langle w, \hat{y}-x\rangle \leqslant 0$ *for all* $x \in S(\hat{y})$.

Proof. By virtue of Theorem 1, we need only show that

$$\Sigma_1 := \{ y \in X : \sup_{x \in S(y)} \sup_{u \in T(x)} \mathrm{Re}\langle u, y-x\rangle > 0\}$$

is open in X. Let $y_0 \in \Sigma_1$; then there exist $x_0 \in S(y_0)$ and $f_0 \in T(x_0)$ such that

$$\alpha := \mathrm{Re}\langle f_0, y_0 - x_0\rangle > 0.$$

Since f_0 is continuous at x_0 and at y_0, there exist an open neighborhood N_1 of x_0 and an open neighborhood U_1 of y_0 such that

$$x \in N_1 \Rightarrow |\langle f_0, x_0\rangle - \langle f_0, x\rangle| < \alpha/6.$$
$$y \in U_1 \Rightarrow |\langle f_0, y_0\rangle - \langle f_0, y\rangle| < \alpha/6.$$

Let

$$W := \{f \in E' : \sup_{z_1, z_2 \in X} |\langle f - f_0, z_1 - z_2\rangle| < \alpha/6\};$$

then W is a strongly open neighborhood of f_0 and $W \cap T(x_0) \neq \emptyset$ so that by lower semicontinuity of T at x_0, there exists an open neighborhood N_2 of x_0 such that

$$x \in N_2 \Rightarrow T(x) \cap W \neq \emptyset.$$

Let $N := N_1 \cap N_2$; since N is a neighborhood of x_0 and $N \cap S(y_0) \neq \emptyset$, by lower semicontinuity of S at y_0, there exists an open neighborhood U_2 of y_0 such that

$$y \in U_2 \Rightarrow S(y) \cap N \neq \emptyset.$$

Let $U := U_1 \cap U_2$; then U is an open neighborhood of y_0. For each $y_1 \in U$, choose $x_1 \in S(y_1) \cap N$ and $f_1 \in T(x_1) \cap W$; it follows that

$$\begin{aligned}
\alpha &= \mathrm{Re}\langle f_0, y_0 - x_0\rangle \\
&= \mathrm{Re}\langle f_1, y_1 - x_1\rangle + \mathrm{Re}\langle f_0, y_0 - y_1\rangle + \mathrm{Re}\langle f_0 - f_1, y_1 - x_1\rangle \\
&\quad + \mathrm{Re}\langle f_0, x_1 - x_0\rangle \\
&< \mathrm{Re}\langle f_1, y_1 - x_1\rangle + \alpha/2.
\end{aligned}$$

Thus, $\mathrm{Re}\langle f_1, y_1 - x_1\rangle > \alpha/2 > 0$ so that $y_1 \in \Sigma_1$ for all $y_1 \in U$. Hence Σ_1 is open in X and the proof is completed. ∎

When $S(x) \equiv X$, Theorem 2 gives a multivalued version of the Hartman–Stampacchia variational inequality [9] as follows.

COROLLARY 2. *Let E be a locally convex Hausdorff topological vector space and X be a nonempty compact convex subset of E. Let $T: X \to 2^{E'}$ be monotone such that for each $x \in X$, $T(x)$ is a nonempty subset of E' and T is lower semicontinuous from the relative topology of X to the strong topology of E'. Then there exists a point $\hat{y} \in X$ such that*

$$\sup_{w \in T(\hat{y})} \mathrm{Re}\langle w, \hat{y} - x\rangle \leqslant 0 \qquad \textit{for all } x \in X.$$

3

In Theorems 1 and 2, T is assumed to be monotone together with some kind of lower semicontinuity. In this section we shall establish results for upper semicontinuous map T without monotonicity.

THEOREM 3. *Let E be a locally convex Hausdorff topological vector space and X be a nonempty compact convex subset of E. Let $S: X \to 2^X$ be upper semicontinuous such that for each $x \in X$, $S(x)$ is a nonempty closed convex subset of X, and let $T: X \to 2^{E'}$ be upper semicontinuous from the relative topology of X to the strong topology of E' such that for each $x \in X$, $T(x)$ is a nonempty compact convex subset of E'. Suppose further that the set $\Sigma_2 := \{ y \in X: \sup_{x \in S(y)} \inf_{z \in T(y)} \operatorname{Re}\langle z, y-x \rangle > 0\}$ is open in X. Then there exists a point $\hat{y} \in X$ such that*

(i) *$\hat{y} \in S(\hat{y})$ and*

(ii) *there exists a point $\hat{z} \in T(\hat{y})$ with $\operatorname{Re}\langle \hat{z}, \hat{y}-x \rangle \leqslant 0$ for all $x \in S(\hat{y})$.*

Proof. We divide the proof into two steps:

Step 1. *There exists a point $\hat{y} \in S$ such that $\hat{y} \in S(\hat{y})$ and* $\sup_{x \in S(\hat{y})} \inf_{z \in T(\hat{y})} \operatorname{Re}\langle z, \hat{y}-x \rangle \leqslant 0$.

Suppose the assertion were false. Then for all $y \in X$, either $y \notin S(y)$ or there exists $x \in S(y)$ such that $\inf_{z \in T(y)} \operatorname{Re}\langle z, y-x \rangle > 0$. Observe that whenever $y \notin S(y)$, there exists $p \in E'$ with

$$\operatorname{Re}\langle p, y \rangle - \sup_{x \in S(y)} \operatorname{Re}\langle p, x \rangle > 0.$$

For each $y \in X$, we set

$$\alpha(y) := \sup_{x \in S(y)} \inf_{z \in T(y)} \operatorname{Re}\langle z, y-x \rangle.$$

Let

$$V_0 := \{ y \in X: \alpha(y) > 0\},$$

and for each $p \in E'$, we set

$$V(p) := \{ y \in X: \operatorname{Re}\langle p, y \rangle - \sup_{x \in S(y)} \operatorname{Re}\langle p, x \rangle > 0\}.$$

Then $X = V_0 \cup \bigcup_{p \in E'} V(p)$. By hypothesis, V_0 is open in X. By Lemma 1, $V(p)$ is open in X for each $p \in E'$. Since X is compact, there exist $p_1, ..., p_n \in E'$ such that

$$X = V_0 \cup \bigcup_{i=1}^{n} V(p_i)$$

and a continuous partition of unity $\{\beta_0, \beta_1, ..., \beta_n\}$ subordinated to the covering $\{V_0, V(p_1), ..., V(p_n)\}$.

Define $\phi: X \times X \to \mathbb{R}$ by setting

$$\phi(x, y) := \beta_0(y) \inf_{w \in T(y)} \operatorname{Re}\langle w, y-x\rangle + \sum_{i=1}^{n} \beta_i(y) \operatorname{Re}\langle p_i, y-x\rangle.$$

Clearly $\phi(x, x) = 0$ for each $x \in X$. Note that for each fixed $x \in X$, $y \to \inf_{w \in T(y)} \operatorname{Re}\langle w, y-x\rangle$ is lower semicontinuous as can be seen within the proof of Theorem 21 in [13], so that $y \to \phi(x, y)$ is lower semicontinuous. Also it is clear that for each fixed $y \in X$, $x \to \phi(x, y)$ is quasi-concave. Hence by the Ky Fan minimax principle (i.e., Theorem A with $\phi \equiv \psi$), there exists a point $\hat{y} \in X$ such that $\phi(x, \hat{y}) \leqslant 0$ for all $x \in X$. The contradiction that there is a point $\hat{x} \in X$ with $\phi(\hat{x}, \hat{y}) > 0$ can be achieved by using the corresponding proof of Step 1 of Theorem 1.

Step 2. *There exists a point* $\hat{z} \in T(\hat{y})$ *such that* $\operatorname{Re}\langle \hat{z}, \hat{y} - x\rangle \leqslant 0$ *for all* $x \in S(\hat{y})$.

Indeed, define $f: S(\hat{y}) \times T(\hat{y}) \to \mathbb{R}$ by

$$f(x, z) := \operatorname{Re}\langle z, \hat{y} - x\rangle.$$

Note that for each fixed $x \in S(\hat{y})$, $z \to f(x, z)$ is continuous and affine, and for each $z \in T(\hat{y})$, $x \to f(x, z)$ is affine. Thus by Kneser's minimax theorem [10], we have

$$\min_{z \in T(\hat{y})} \max_{x \in S(\hat{y})} f(x, z) = \max_{x \in S(\hat{y})} \min_{z \in T(\hat{y})} f(x, z).$$

Thus

$$\min_{z \in T(\hat{y})} \max_{x \in S(\hat{y})} \operatorname{Re}\langle z, \hat{y} - x\rangle \leqslant 0 \qquad \text{by Step 1.}$$

Since $T(\hat{y})$ is compact, there exists $\hat{z} \in T(\hat{y})$ such that

$$\operatorname{Re}\langle \hat{z}, \hat{y} - x\rangle \leqslant 0 \qquad \text{for all } x \in S(\hat{y}). \quad \blacksquare$$

When E is a normed linear space, by imposing additional lower semicontinuity on S, the interacting set Σ_2 in Theorem 3 is always open:

THEOREM 4. *Let E be a normed linear space and X be a nonempty compact convex subset of E. Let $S: X \to 2^X$ be continuous such that for each $x \in X$, $S(x)$ is a nonempty closed convex subset of X, and let $T: X \to 2^{E'}$ be upper semicontinuous such that for each $x \in X$, $T(x)$ is a nonempty compact convex subset of E'. Then there exists a point $\hat{y} \in X$ such that*

(i) $\hat{y} \in S(\hat{y})$ *and*

(ii) *there exists a point* $\hat{z} \in T(\hat{y})$ *with* $\mathrm{Re}\langle \hat{z}, \hat{y} - x\rangle \leqslant 0$ *for all* $x \in S(\hat{y})$.

Proof. By virtue of Theorem 3, we need only show that the set

$$\Sigma_2 := \{ y \in X\colon \sup_{x \in S(y)} \inf_{z \in T(y)} \mathrm{Re}\langle z, y - x\rangle > 0\}$$

is open in X. For this purpose, let $y_0 \in \Sigma_2$, then there exists $x_0 \in S(y_0)$ with

$$\alpha = \inf_{z \in T(y_0)} \mathrm{Re}\langle z, y_0 - x_0\rangle > 0.$$

Let

$$M := \max\{\mathrm{diam}(x), \sup_{z \in T(y_0)} \|z\|\} \qquad \text{and} \qquad B := \{f \in E' : \|f\| < 1\}.$$

Since T is upper semicontinuous at y_0, for $\eta = \alpha/6(1+M) > 0$, there exists $\delta_1 \in (0, \min\{1, \alpha/6(1+M)\})$ such that for all $y \in X$, $\|y - y_0\| < \delta_1$ implies $T(y) \subset T(y_0) + \eta B$. As S is lower semicontinuous at y_0, there exists $\delta_2 \in (0, \min\{1, \alpha/6(1+M)\})$ such that for all $y \in X$, $\|y - y_0\| < \delta_2$ implies

$$S(y) \cap \{x \in X\colon \|x - x_0\| < \eta\} \neq \varnothing.$$

Let $\delta := \min\{\delta_1, \delta_2\}$. Let $y_1 \in X$ be such that $\|y_1 - y_0\| < \delta$. Then $T(y_1) \subset T(y_0) + \eta B$ and we can choose $x_1 \in S(y_1)$ with $\|x_1 - x_0\| < \eta$. It follows that

$$\begin{aligned}
&\inf_{z \in T(y_1)} \mathrm{Re}\langle z, y_1 - x_1\rangle \\
&\quad\geqslant \inf_{Z \in T(y0) + \eta B} \mathrm{Re}\langle z, y_1 - x_1\rangle \\
&\quad\geqslant \inf_{z \in T(y_0)} \mathrm{Re}\langle z, y_1 - x_1\rangle + \inf_{z \in \eta B} \mathrm{Re}\langle z, y_1 - x_1\rangle \\
&\quad\geqslant \inf_{z \in T(y_0)} \mathrm{Re}\langle z, y_1 - y_0\rangle + \inf_{z \in T(y_0)} \mathrm{Re}\langle z, y_0 - x_0\rangle \\
&\qquad + \inf_{z \in T(y_0)} \mathrm{Re}\langle z, x_0 - x_1\rangle - \eta\,\|y_1 - x_1\| \\
&\quad\geqslant -\sup_{z \in T(y_0)} \|z\|\,\|y_1 - y_0\| + \alpha \\
&\qquad - \sup_{z \in T(y_0)} \|z\|\,\|x_0 - x_1\| - \alpha/6 \\
&\quad> \alpha/2 > 0.
\end{aligned}$$

Thus,

$$\sup_{x \in S(y_1)} \inf_{z \in T(y_1)} \mathrm{Re}\langle z, y_1 - x\rangle > 0$$

so that $y_1 \in \Sigma_2$ whenever $y_1 \in X$ with $\| y_1 - y_0 \| < \delta$. This shows that Σ_2 is open in X and the proof is completed. ∎

When $S(x) \equiv X$, we obtain another multivalued version of the Hartman–Stampacchia variational inequality as follows.

COROLLARY 3. *Let E be a normed linear space and $X \subset E$ a nonempty compact convex subset of E. Let $T: X \to 2^{E'}$ be upper semicontinuous such that for each $x \in X$, $T(x)$ is a nonempty compact convex subset of E'. Then there exist a point $\hat{y} \in X$ and a point $\hat{z} \in T(\hat{y})$ such that*

$$\operatorname{Re}\langle \hat{z}, \hat{y} - x \rangle \leqslant 0 \qquad \textit{for all } x \in X.$$

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